

Convergence Rates in Parabolic Homogenization with Time-Dependent Periodic Coefficients

Jun Geng^{*} Zhongwei Shen[†]

Abstract

For a family of second-order parabolic systems with bounded measurable, rapidly oscillating and time-dependent periodic coefficients, we investigate the sharp convergence rates of weak solutions in L^2 . Both initial-Dirichlet and initial-Neumann problems are studied.

1 Introduction

The primary purpose of this paper is to investigate the sharp convergence rates in L^2 for a family of second-order parabolic operators $\partial_t + \mathcal{L}_\varepsilon$ with bounded measurable, rapidly oscillating and time-dependent periodic coefficients. Both the initial-Dirichlet and initial-Neumann boundary value problems are studied. Specifically, we consider

$$\mathcal{L}_\varepsilon = -\operatorname{div} \left(A(x/\varepsilon, t/\varepsilon^2) \nabla \right), \quad (1.1)$$

where $\varepsilon > 0$ and $A(y, s) = (a_{ij}^{\alpha\beta}(y, s))$ with $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. Throughout this paper we will assume that the coefficient matrix $A(y, s)$ is real, bounded measurable, and satisfies the ellipticity condition,

$$\mu |\xi|^2 \leq a_{ij}^{\alpha\beta}(y, s) \xi_i^\alpha \xi_j^\beta \leq \frac{1}{\mu} |\xi|^2 \quad \text{for any } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d} \text{ and a.e. } (y, s) \in \mathbb{R}^{d+1}, \quad (1.2)$$

where $\mu > 0$, and the periodicity condition,

$$A(y + z, s + t) = A(y, s) \quad \text{for } (z, t) \in \mathbb{Z}^{d+1} \text{ and a.e. } (y, s) \in \mathbb{R}^{d+1}. \quad (1.3)$$

No additional smoothness condition will be imposed on A .

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $0 < T < \infty$. We are interested in the initial-Dirichlet problem,

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F & \text{in } \Omega \times (0, T), \\ u_\varepsilon = g & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon = h & \text{on } \Omega \times \{t = 0\}, \end{cases} \quad (1.4)$$

^{*}Supported in part by the NNSF of China (11571152) and Fundamental Research Funds for the Central Universities (LZUJBKY-2015-72).

[†]Supported in part by NSF grant DMS-1161154.

and the initial-Neumann problem,

$$\begin{cases} (\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F & \text{in } \Omega \times (0, T), \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega \times (0, T), \\ u_\varepsilon = h & \text{on } \Omega \times \{t = 0\}, \end{cases} \quad (1.5)$$

where $\left(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right)^\alpha = n_i a_{ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \frac{\partial u_\varepsilon^\beta}{\partial x_j}$ denotes the conormal derivative of u_ε associated with \mathcal{L}_ε and $n = (n_1, \dots, n_d)$ is the outward normal to $\partial\Omega$. Under suitable conditions on F , g , h and Ω , it is known that the weak solution u_ε of (1.4) converges weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(\Omega_T)$ to u_0 , where $\Omega_T = \Omega \times (0, T)$. Furthermore, the function u_0 is the weak solution of the (homogenized) initial-Dirichlet problem,

$$\begin{cases} (\partial_t + \mathcal{L}_0)u_0 = F & \text{in } \Omega \times (0, T), \\ u_0 = g & \text{on } \partial\Omega \times (0, T), \\ u_0 = h & \text{on } \Omega \times \{t = 0\}. \end{cases} \quad (1.6)$$

Similarly, the weak solution u_ε of (1.5) converges weakly in $L^2(0, T; H^1(\Omega))$ and strongly in $L^2(\Omega_T)$ to the weak solution of the (homogenized) initial-Neumann problem,

$$\begin{cases} (\partial_t + \mathcal{L}_0)u_0 = F & \text{in } \Omega \times (0, T), \\ \frac{\partial u_0}{\partial \nu_0} = g & \text{on } \partial\Omega \times (0, T), \\ u_0 = h & \text{on } \Omega \times \{t = 0\}. \end{cases} \quad (1.7)$$

The operator \mathcal{L}_0 in (1.6) and (1.7), called the homogenized operator, is a second-order elliptic operator with constant coefficients [4].

The following are the main results of the paper, which establish the sharp $O(\varepsilon)$ convergence rates in $L^2(\Omega_T)$ for both the initial-Dirichlet and the initial-Neumann problems.

Theorem 1.1. *Suppose that the coefficient matrix A satisfies (1.2) and (1.3). Let Ω be a bounded $C^{1,1}$ domain in \mathbb{R}^d . Let $u_\varepsilon, u_0 \in L^2(0, T; H^1(\Omega))$ be weak solutions of (1.4) and (1.6), respectively, for some $F \in L^2(\Omega_T)$. Assume that $u_0 \in L^2(0, T; H^2(\Omega))$. Then*

$$\begin{aligned} & \|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \\ & \leq C\varepsilon \left\{ \|u_0\|_{L^2(0, T; H^2(\Omega))} + \|F\|_{L^2(\Omega_T)} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_\Omega |\nabla u_0|^2 \right)^{1/2} \right\}, \end{aligned} \quad (1.8)$$

where C depends at most on d , m , μ , T and Ω .

Theorem 1.2. *Let $u_\varepsilon \in L^2(0, T; H^1(\Omega))$ be a weak solution of (1.5) for some $F \in L^2(\Omega_T)$ and $u_0 \in L^2(0, T; H^1(\Omega))$ the weak solution of the homogenized problem (1.7). Under the same assumptions as in Theorem 1.1, the estimate (1.8) holds.*

Remark 1.3. In Theorems 1.1 and 1.2 we do not specify the conditions directly on g and h , but rather require $u_0 \in L^2(0, T; H^2(\Omega))$. In the case that either $u_\varepsilon = u_0 = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \frac{\partial u_0}{\partial \nu_0} = 0$ on $\partial\Omega \times (0, T)$, i.e. $g = 0$, the third term in the r.h.s. of (1.8) may be bounded by

$$C\{\|\partial_t u_0\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} + \|h\|_{L^2(\Omega)}\}.$$

See (3.19). As a result, we obtain

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \leq C\varepsilon\left\{\|u_0\|_{L^2(0,T;H^2(\Omega))} + \|F\|_{L^2(\Omega_T)} + \|h\|_{L^2(\Omega)}\right\}, \quad (1.9)$$

where C depends at most on d, m, μ, T and Ω . In particular, if $g = 0$ and $h = 0$, then

$$\|u_0\|_{L^2(0,T;H^2(\Omega))} \leq C\|F\|_{L^2(\Omega_T)}$$

(see (3.22)). It follows that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \leq C\varepsilon\|F\|_{L^2(\Omega_T)}. \quad (1.10)$$

Also, in the case that $g = 0$ on $\partial\Omega \times (0, T)$ and $h \in H^1(\Omega)$, it is known that if $\mathcal{L}_0^* = \mathcal{L}_0$, then

$$\|u_0\|_{L^2(0,T;H^2(\Omega))} \leq C\left\{\|F\|_{L^2(\Omega_T)} + \|h\|_{H^1(\Omega)}\right\}$$

[13]. This gives

$$\|u_\varepsilon - u_0\|_{L^2(\Omega_T)} \leq C\varepsilon\left\{\|F\|_{L^2(\Omega_T)} + \|h\|_{H^1(\Omega)}\right\}, \quad (1.11)$$

where C depends at most on d, m, μ, T and Ω .

The sharp convergence rate is one of the central issues in quantitative homogenization and has been studied extensively in the various settings. For elliptic equations and systems in divergence form with periodic coefficients, related results may be found in the recent work [18, 19, 10, 12, 11, 15, 8, 16] (also see [4, 9, 6, 7, 14] for references on earlier work). In particular, the order sharp estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon\|F\|_{L^2(\Omega)}, \quad (1.12)$$

holds, if $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0) = F$ in Ω and $u_\varepsilon = u_0 = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \frac{\partial u_0}{\partial \nu_0} = 0$ on $\partial\Omega$ (see [18, 19, 8, 16] for $C^{1,1}$ domains and [10, 11, 15] for Lipschitz domains). For parabolic equations and systems various results are known in the case where the coefficients are time-independent [9, 17, 21, 20]. We note that in this case, using the partial Fourier transform in the t variable, it is possible to represent the solution of the parabolic system as an integral of the resolvent of the elliptic operator \mathcal{L}_ε and apply the elliptic estimates.

Very few results are known if the coefficients are time-dependent. In fact, to the authors' best knowledge, the only known estimate in this case is

$$\|u_\varepsilon - u_0\|_{L^\infty(\Omega_T)} \leq C\varepsilon, \quad (1.13)$$

obtained by the use of the maximum principle, where C depends on u_0 and coefficients are assumed to be smooth [4]. Our order sharp estimates (1.9)-(1.11), which extend (1.12) to the parabolic setting, seem to be the first work in this area beyond the rough estimate (1.13).

We now describe some of key ideas in the proof of Theorems 1.1 and 1.2. Although it is not clear how to reduce parabolic systems with time-dependent coefficients to elliptic systems by some simple transformations, our general approach to the estimate (1.8) is inspired by the work on elliptic systems mentioned above. We consider the function

$$w_\varepsilon = u_\varepsilon(x, t) - u_0(x, t) - \varepsilon \chi(x/\varepsilon, t/\varepsilon^2) K_\varepsilon(\nabla u_0) - \varepsilon^2 \phi(x/\varepsilon, t/\varepsilon^2) \nabla K_\varepsilon(\nabla u_0), \quad (1.14)$$

where $\chi(y, s)$ and $\phi(y, s)$ are correctors and dual correctors for the family of operators $\partial_t + \mathcal{L}_\varepsilon$, $\varepsilon > 0$ (see Section 2 for their definitions). In (1.14) the operator $K_\varepsilon : L^2(\Omega_T) \rightarrow C_0^\infty(\Omega_T)$ is a parabolic smoothing operator at scale ε . We note that in the elliptic case [18, 19, 15, 16], only the first three terms in the r.h.s. of (1.14) are used. By computing $(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon$, we are able to show that

$$\begin{aligned} & \left| \int_0^T \langle \partial_t w_\varepsilon, \psi \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \right| \\ & \leq C \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,\varepsilon})} \right\} \\ & \quad \cdot \left\{ \varepsilon \|\nabla \psi\|_{L^2(\Omega_T)} + \varepsilon^{1/2} \|\nabla \psi\|_{L^2(\Omega_{T,\varepsilon})} \right\} \end{aligned} \quad (1.15)$$

for any $\psi \in L^2(0, T; H_0^1(\Omega))$ in the case of Dirichlet condition (1.4), and for any $\psi \in L^2(0, T; H^1(\Omega))$ in the case of the Neumann condition (1.5), where $\Omega_{T,\varepsilon}$ denotes the set of points in Ω_T whose (parabolic) distances to the boundary of Ω_T are less than ε (see Section 3 for details). By taking $\psi = w_\varepsilon$ in (1.15) we obtain an $O(\sqrt{\varepsilon})$ error estimate in $L^2(0, T; H^1(\Omega))$,

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \leq C \sqrt{\varepsilon} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,\varepsilon})} \right\}, \quad (1.16)$$

which is more or less sharp, for both the initial-Dirichlet and the initial-Neumann problems. Finally, with (1.15) at our disposal, we give the proof of Theorems 1.1 and 1.2 in Section 4. This is done by a dual argument, inspired by [18, 19].

We point out that results on convergence rates are useful in the study of regularity estimates that are uniform in $\varepsilon > 0$ [2, 1, 15]. For solutions of $(\partial_t + \mathcal{L}_\varepsilon)u_\varepsilon = F$, the uniform boundary Hölder and interior Lipschitz estimates were proved in [5] by a compactness method, introduced to the study of homogenization problems in [3]. The results obtained in this paper should allow us to establish the boundary Lipschitz estimates as well as Rellich estimates at large scale for parabolic systems in a manner similar to that in [15] for elliptic systems of linear elasticity. We plan to carry this out in a separate study.

We end this section with some notations that will be used throughout the paper. A function $h = h(y, s)$ in \mathbb{R}^{d+1} is said to be 1-periodic if h is periodic with respect to \mathbb{Z}^{d+1} . We will use the notation

$$h^\varepsilon(x, t) = h(x/\varepsilon, t/\varepsilon^2)$$

for $\varepsilon > 0$, and the summation convention that the repeated indices are summed. Finally, we use C to denote constants that depend at most on d, m, μ, T and Ω , but never on ε .

2 Correctors and dual correctors

Let $\mathcal{L}_\varepsilon = -\operatorname{div}(A^\varepsilon(x, t)\nabla)$, where $A^\varepsilon(x, t) = A(x/\varepsilon, t/\varepsilon^2)$ and $A(y, s)$ is 1-periodic and satisfies the ellipticity condition (1.2). For $1 \leq j \leq d$ and $1 \leq \beta \leq m$, the corrector $\chi_j^\beta = \chi_j^\beta(y, s) = (\chi_j^{\alpha\beta}(y, s))$ is defined as the weak solution of the following cell problem:

$$\begin{cases} (\partial_s + \mathcal{L}_1)(\chi_j^\beta) = -\mathcal{L}_1(P_j^\beta) & \text{in } Y, \\ \chi_j^\beta = \chi_j^\beta(y, s) & \text{is 1-periodic in } (y, s), \\ \int_Y \chi_j^\beta = 0, \end{cases} \quad (2.1)$$

where $Y = [0, 1)^{d+1}$, $P_j^\beta(y) = y_j e^\beta$, and $e^\beta = (0, \dots, 1, \dots, 0)$ with 1 in the β^{th} position. Note that

$$(\partial_s + \mathcal{L}_1)(\chi_j^\beta + P_j^\beta) = 0 \quad \text{in } \mathbb{R}^{d+1}. \quad (2.2)$$

By the rescaling property of $\partial_t + \mathcal{L}_\varepsilon$, one obtains that

$$(\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon \chi_j^\beta(x/\varepsilon, t/\varepsilon^2) + P_j^\beta(x) \right\} = 0 \quad \text{in } \mathbb{R}^{d+1}. \quad (2.3)$$

Let $\hat{A} = (\hat{a}_{ij}^{\alpha\beta})$, where $1 \leq i, j \leq d$, $1 \leq \alpha, \beta \leq m$, and

$$\hat{a}_{ij}^{\alpha\beta} = \oint_Y \left[a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k} \chi_j^{\gamma\beta} \right]; \quad (2.4)$$

that is

$$\hat{A} = \oint_Y \{ A + A \nabla \chi \}.$$

It is known that the constant matrix \hat{A} satisfies the ellipticity condition,

$$\mu |\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \leq \mu_1 |\xi|^2 \quad \text{for any } \xi = (\xi_j^\beta) \in \mathbb{R}^{m \times d},$$

where $\mu_1 > 0$ depends only on d, m and μ [4]. Denote $\mathcal{L}_0 = -\operatorname{div}(\hat{A}\nabla)$. Then $\partial_t + \mathcal{L}_0$ is the homogenized operator for the family of parabolic operators $\partial_t + \mathcal{L}_\varepsilon$, $\varepsilon > 0$.

To introduce the dual correctors, we consider the 1-periodic matrix-valued function

$$B = A + A \nabla \chi - \hat{A}. \quad (2.5)$$

More precisely, $B = B(y, s) = (b_{ij}^{\alpha\beta})$, where $1 \leq i, j \leq d$, $1 \leq \alpha, \beta \leq m$, and

$$b_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} - \hat{a}_{ij}^{\alpha\beta}. \quad (2.6)$$

Lemma 2.1. *Let $1 \leq j \leq d$ and $1 \leq \alpha, \beta \leq m$. Then there exist 1-periodic functions $\phi_{kij}^{\alpha\beta}(y, s)$ in \mathbb{R}^{d+1} such that $\phi_{kij}^{\alpha\beta} \in H^1(Y)$,*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} (\phi_{kij}^{\alpha\beta}) \quad \text{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}, \quad (2.7)$$

where $1 \leq k, i \leq d+1$, $b_{ij}^{\alpha\beta}$ is defined by (2.6) for $1 \leq i \leq d$, $b_{(d+1)j}^{\alpha\beta} = -\chi_j^{\alpha\beta}$, and we have used the notation $y_{d+1} = s$.

Proof. Observe that by (2.1) and (2.4), $b_{ij}^{\alpha\beta} \in L^2(Y)$ and

$$\int_Y b_{ij}^{\alpha\beta} = 0 \quad (2.8)$$

for $1 \leq i \leq d+1$. It follows that there exist $f_{ij}^{\alpha\beta} \in H^2(Y)$ such that

$$\begin{cases} \Delta_{d+1} f_{ij}^{\alpha\beta} = b_{ij}^{\alpha\beta} & \text{in } \mathbb{R}^{d+1}, \\ f_{ij}^{\alpha\beta} \text{ is 1-periodic} & \text{in } \mathbb{R}^{d+1}, \end{cases} \quad (2.9)$$

where Δ_{d+1} denotes the Laplacian in \mathbb{R}^{d+1} . Write

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} \left\{ \frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta} \right\} + \frac{\partial}{\partial y_i} \left\{ \frac{\partial}{\partial y_k} f_{kj}^{\alpha\beta} \right\}, \quad (2.10)$$

where the index k is summed from 1 to $d+1$. Note that by (2.1),

$$\sum_{i=1}^{d+1} \frac{\partial b_{ij}^{\alpha\beta}}{\partial y_i} = \sum_{i=1}^d \frac{\partial}{\partial y_i} b_{ij}^{\alpha\beta} - \frac{\partial}{\partial s} \chi_j^{\alpha\beta} = 0. \quad (2.11)$$

In view of (2.9) this implies that

$$\sum_{i=1}^{d+1} \frac{\partial}{\partial y_i} f_{ij}^{\alpha\beta}$$

is harmonic in \mathbb{R}^{d+1} . Since it is 1-periodic, it must be constant. Consequently, by (2.10), we obtain

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k} (\phi_{kij}^{\alpha\beta}), \quad (2.12)$$

where

$$\phi_{kij}^{\alpha\beta} = \frac{\partial}{\partial y_k} f_{ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} f_{kj}^{\alpha\beta} \quad (2.13)$$

is 1-periodic and belongs to $H^1(Y)$. It is easy to see that $\phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}$. This completes the proof. \square

The 1-periodic functions $(\phi_{kij}^{\alpha\beta})$ given by Lemma 2.1 are called dual correctors for the family of parabolic operators $\partial_t + \mathcal{L}_\varepsilon$, $\varepsilon > 0$. As in the elliptic case [9, 10], they play an important role in the study of the problem of convergence rates. Indeed, to establish the main results of this paper, we shall consider the function $w_\varepsilon = (w_\varepsilon^\alpha)$, where

$$\begin{aligned} w_\varepsilon^\alpha(x, t) = & u_\varepsilon^\alpha(x, t) - u_0^\alpha(x, t) - \varepsilon \chi_j^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \\ & - \varepsilon^2 \phi_{(d+1)ij}^{\alpha\beta}(x/\varepsilon, t/\varepsilon^2) \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0^\beta}{\partial x_j} \right), \end{aligned} \quad (2.14)$$

and $K_\varepsilon : L^2(\Omega_T) \rightarrow C_0^\infty(\Omega_T)$ is a linear operator to be chosen later. The repeated indices i, j in (2.14) are summed from 1 to d .

Theorem 2.2. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $0 < T < \infty$. Let $u_\varepsilon \in L^2(0, T; H^1(\Omega))$ and $u_0 \in L^2(0, T; H^2(\Omega))$ be solutions of the initial-Dirichlet problems (1.4) and (1.6), respectively. Let w_ε be defined by (2.14). Then for any $\psi \in L^2(0, T; H_0^1(\Omega))$,

$$\begin{aligned}
& \int_0^T \langle \partial_t w_\varepsilon, \psi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \\
&= \iint_{\Omega_T} (\widehat{a}_{ij} - a_{ij}^\varepsilon) \left(\frac{\partial u_0}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right) \frac{\partial \psi}{\partial x_i} \\
&\quad - \varepsilon \iint_{\Omega_T} a_{ij}^\varepsilon \cdot \chi_k^\varepsilon \cdot \frac{\partial}{\partial x_j} K_\varepsilon \left(\frac{\partial u_0}{\partial x_k} \right) \cdot \frac{\partial \psi}{\partial x_i} \\
&\quad - \varepsilon \iint_{\Omega_T} \phi_{kij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \cdot \frac{\partial \psi}{\partial x_k} \\
&\quad - \varepsilon^2 \iint_{\Omega_T} \phi_{k(d+1)j}^\varepsilon \cdot \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \cdot \frac{\partial \psi}{\partial x_k} \\
&\quad + \varepsilon \iint_{\Omega_T} a_{ij}^\varepsilon \cdot \left(\frac{\partial}{\partial x_j} (\phi_{(d+1)\ell k}) \right)^\varepsilon \cdot \frac{\partial}{\partial x_\ell} K_\varepsilon \left(\frac{\partial u_0}{\partial x_k} \right) \cdot \frac{\partial \psi}{\partial x_i} \\
&\quad + \varepsilon^2 \iint_{\Omega_T} a_{ij}^\varepsilon \cdot \phi_{(d+1)\ell k}^\varepsilon \cdot \frac{\partial^2}{\partial x_j \partial x_\ell} K_\varepsilon \left(\frac{\partial u_0}{\partial x_k} \right) \cdot \frac{\partial \psi}{\partial x_i},
\end{aligned} \tag{2.15}$$

where we have suppressed superscripts α, β for the simplicity of presentation. The repeated indices i, j, k, ℓ are summed from 1 to d .

Proof. Using (1.4) and (1.6), we see that

$$\begin{aligned}
(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon &= (\mathcal{L}_0 - \mathcal{L}_\varepsilon)u_0 - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon \chi_j^\varepsilon K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} \\
&\quad - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} \\
&= -\frac{\partial}{\partial x_i} \left\{ (\widehat{a}_{ij} - a_{ij}^\varepsilon) \left(\frac{\partial u_0}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right) \right\} \\
&\quad - \frac{\partial}{\partial x_i} \left\{ (\widehat{a}_{ij} - a_{ij}^\varepsilon) K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon \chi_j^\varepsilon K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} \\
&\quad - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}.
\end{aligned}$$

By computing the third term in the r.h.s. of the equalities above and using (2.6), we obtain

$$\begin{aligned}
(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon &= -\frac{\partial}{\partial x_i} \left\{ (\widehat{a}_{ij} - a_{ij}^\varepsilon) \left(\frac{\partial u_0}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right) \right\} \\
&\quad + \frac{\partial}{\partial x_i} \left\{ b_{ij}^\varepsilon K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^\varepsilon \cdot \chi_k^\varepsilon \cdot \frac{\partial}{\partial x_j} K_\varepsilon \left(\frac{\partial u_0}{\partial x_k} \right) \right\} \\
&\quad - \varepsilon \partial_t \left\{ \chi_j^\varepsilon K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}.
\end{aligned}$$

In view of (2.11) this gives

$$\begin{aligned}
(\partial_t + \mathcal{L}_\varepsilon)w_\varepsilon = & -\frac{\partial}{\partial x_i} \left\{ (\widehat{a}_{ij} - a_{ij}^\varepsilon) \left(\frac{\partial u_0}{\partial x_j} - K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right) \right\} \\
& + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^\varepsilon \cdot \chi_k^\varepsilon \cdot \frac{\partial}{\partial x_j} K_\varepsilon \left(\frac{\partial u_0}{\partial x_k} \right) \right\} + b_{ij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \\
& - \varepsilon \chi_j^\varepsilon \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) - (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}.
\end{aligned} \tag{2.16}$$

Next, by Lemma 2.1, we may write

$$\begin{aligned}
& b_{ij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) - \varepsilon \chi_j^\varepsilon \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \\
& = \varepsilon \frac{\partial}{\partial x_k} \left(\phi_{kij}^\varepsilon \right) \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) + \varepsilon^2 \partial_t \left(\phi_{(d+1)ij}^\varepsilon \right) \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \\
& \quad + \varepsilon^2 \frac{\partial}{\partial x_k} \left(\phi_{k(d+1)j}^\varepsilon \right) \cdot \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right),
\end{aligned}$$

where we have also used the fact $\phi_{(d+1)(d+1)j} = 0$. Furthermore, by the skew-symmetry in (2.7), we see that

$$\begin{aligned}
& b_{ij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) - \varepsilon \chi_j^\varepsilon \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \\
& = \varepsilon \frac{\partial}{\partial x_k} \left\{ \phi_{kij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} + \varepsilon^2 \partial_t \left\{ \phi_{(d+1)ij}^\varepsilon \cdot \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\} \\
& \quad + \varepsilon^2 \frac{\partial}{\partial x_k} \left\{ \phi_{k(d+1)j}^\varepsilon \cdot \partial_t K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}.
\end{aligned}$$

This, combined with (2.16), gives the desired equation (2.15). \square

The next theorem is concerned with the initial-Neumann problem.

Theorem 2.3. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $0 < T < \infty$. Let $u_\varepsilon \in L^2(0, T; H^1(\Omega))$ and $u_0 \in L^2(0, T; H^2(\Omega))$ be solutions of the initial-Neumann problems (1.5) and (1.7), respectively. Let w_ε be defined by (2.14). Then the equation (2.15) holds for any $\psi \in L^2(0, T; H^1(\Omega))$, if $\langle \cdot, \cdot \rangle$ in its l.h.s. denotes the pairing between $H^1(\Omega)$ and its dual.*

Proof. It follows from (1.5) and (1.7) that

$$\int_0^T \langle \partial_t u_\varepsilon, \psi \rangle + \iint_{\Omega_T} A^\varepsilon \nabla u_\varepsilon \cdot \nabla \psi = \int_0^T \langle \partial_t u_0, \psi \rangle + \iint_{\Omega_T} \widehat{A} \nabla u_0 \cdot \nabla \psi$$

for any $\psi \in L^2(0, T; H^1(\Omega))$. This gives

$$\begin{aligned}
& \int_0^T \langle \partial_t w_\varepsilon, \psi \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \\
& = \iint_{\Omega_T} (\widehat{A} - A^\varepsilon) \nabla u_0 \cdot \nabla \psi - \int_0^T \left\langle (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon \chi_j^\varepsilon K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}, \psi \right\rangle \\
& \quad - \int_0^T \left\langle (\partial_t + \mathcal{L}_\varepsilon) \left\{ \varepsilon^2 \phi_{(d+1)ij}^\varepsilon \frac{\partial}{\partial x_i} K_\varepsilon \left(\frac{\partial u_0}{\partial x_j} \right) \right\}, \psi \right\rangle,
\end{aligned}$$

where we have used the fact $K_\varepsilon(\nabla u_0) \in C_0^\infty(\Omega_T)$. The rest of the proof is similar to that of Theorem 2.2. We omit the details. \square

3 Error estimates in $L^2(0, T; H^1(\Omega))$

We begin by introducing a parabolic smoothing operator. Fix a nonnegative function $\theta = \theta(y, s) \in C_0^\infty(B(0, 1))$ such that $\int_{\mathbb{R}^{d+1}} \theta = 1$. Define

$$\begin{aligned} S_\varepsilon(f)(x, t) &= \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^{d+1}} f(x - y, t - s) \theta(y/\varepsilon, s/\varepsilon^2) dy ds \\ &= \int_{\mathbb{R}^{d+1}} f(x - \varepsilon y, t - \varepsilon^2 s) \theta(y, s) dy ds. \end{aligned} \quad (3.1)$$

Lemma 3.1. *Let S_ε be defined as in (3.1). Then*

$$\|S_\varepsilon(f)\|_{L^2(\mathbb{R}^{d+1})} \leq \|f\|_{L^2(\mathbb{R}^{d+1})}, \quad (3.2)$$

$$\varepsilon \|\nabla S_\varepsilon(f)\|_{L^2(\mathbb{R}^{d+1})} + \varepsilon^2 \|\nabla^2 S_\varepsilon(f)\|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(\mathbb{R}^{d+1})}, \quad (3.3)$$

$$\varepsilon^2 \|\partial_t S_\varepsilon(f)\|_{L^2(\mathbb{R}^{d+1})} \leq C \|f\|_{L^2(\mathbb{R}^{d+1})}, \quad (3.4)$$

where C depends only on d .

Proof. This follows easily from the Plancherel Theorem. \square

Lemma 3.2. *Let S_ε be defined as in (3.1). Then*

$$\|\nabla S_\varepsilon(f) - \nabla f\|_{L^2(\mathbb{R}^{d+1})} \leq C\varepsilon \left\{ \|\nabla^2 f\|_{L^2(\mathbb{R}^{d+1})} + \|\partial_t f\|_{L^2(\mathbb{R}^{d+1})} \right\}, \quad (3.5)$$

where C depends only on d .

Proof. By the Plancherel Theorem it suffices to show that

$$|\xi_i \widehat{\theta}(\varepsilon \xi', \varepsilon^2 \xi_{d+1}) - \xi_i \widehat{\theta}(0, 0)| \leq C\varepsilon \{ |\xi'|^2 + |\xi_{d+1}| \},$$

where $1 \leq i \leq d$ and $\xi' = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Furthermore, by a change of variables, one may assume that $\varepsilon = 1$. In this case, if $|\xi'| \geq 1$, then

$$|\xi_i \widehat{\theta}(\xi', \xi_{d+1}) - \xi_i \widehat{\theta}(0, 0)| \leq C |\xi'| \leq C(|\xi'|^2 + |\xi_{d+1}|).$$

If $|\xi'| \leq 1$, we have

$$|\xi_i \widehat{\theta}(\xi', \xi_{d+1}) - \xi_i \widehat{\theta}(0, 0)| \leq C |\xi'| (|\xi'| + |\xi_{d+1}|) \leq C(|\xi'|^2 + |\xi_{d+1}|).$$

This completes the proof. \square

Lemma 3.3. *Let $g = g(y, s)$ be a 1-periodic function in (y, s) . Then*

$$\|g^\varepsilon S_\varepsilon(f)\|_{L^p(\mathbb{R}^{d+1})} \leq C \|g\|_{L^p(Y)} \|f\|_{L^p(\mathbb{R}^{d+1})} \quad (3.6)$$

for any $1 \leq p < \infty$, where $g^\varepsilon(x, t) = g(x/\varepsilon, t/\varepsilon^2)$ and C depends only on d and p .

Proof. Note that $S_\varepsilon(f)(x, t) = S_1(f_\varepsilon)(\varepsilon^{-1}x, \varepsilon^{-2}t)$, where $f_\varepsilon(x, t) = f(\varepsilon x, \varepsilon^2 t)$. As a result, by a change of variables, it suffices to consider the case $\varepsilon = 1$. In this case we first use $\int_{\mathbb{R}^{d+1}} \theta = 1$ and Hölder's inequality to obtain

$$|S_1(f)(x, t)|^p \leq \int_{\mathbb{R}^{d+1}} |f(y, s)|^p \theta(x - y, t - s) dy ds.$$

It follows by Fubini's Theorem that

$$\begin{aligned} \int_{\mathbb{R}^{d+1}} |g(x, t)|^p |S_1(f)(x, t)|^p dx dt &\leq \sup_{(y, s) \in \mathbb{R}^{d+1}} \int_{B((y, s), 1)} |g(x, t)|^p dx dt \int_{\mathbb{R}^{d+1}} |f(y, s)|^p dy ds \\ &\leq C \|g\|_{L^p(Y)}^p \|f\|_{L^p(\mathbb{R}^{d+1})}^p, \end{aligned}$$

where C depends only on d . This gives (3.6) for the case $\varepsilon = 1$. \square

Remark 3.4. The same argument as in the proof of Lemma 3.3 also shows that

$$\begin{aligned} \|g^\varepsilon \nabla S_\varepsilon(f)\|_{L^p(\mathbb{R}^{d+1})} &\leq C \varepsilon^{-1} \|g\|_{L^p(Y)} \|f\|_{L^p(\mathbb{R}^{d+1})}, \\ \|g^\varepsilon \partial_t S_\varepsilon(f)\|_{L^p(\mathbb{R}^{d+1})} &\leq C \varepsilon^{-2} \|g\|_{L^p(Y)} \|f\|_{L^p(\mathbb{R}^{d+1})} \end{aligned} \quad (3.7)$$

for $1 \leq p < \infty$, where C depends only on d and p .

Let $\delta \in (2\varepsilon, 20\varepsilon)$. Choose $\eta_1 \in C_0^\infty(\Omega)$ such that $0 \leq \eta_1 \leq 1$, $\eta_1(x) = 1$ if $\text{dist}(x, \partial\Omega) \geq 2\delta$, $\eta_1(x) = 0$ if $\text{dist}(x, \partial\Omega) \leq \delta$, and $|\nabla_x \eta_1| \leq C\delta^{-1}$. Similarly, we choose $\eta_2 \in C_0^\infty(0, T)$ such that $0 \leq \eta_2 \leq 1$, $\eta_2(t) = 1$ if $2\delta^2 \leq t \leq T - 2\delta^2$, $\eta_2(t) = 0$ if $t \leq \delta^2$ or $t > T - \delta^2$, and $|\eta_2'(t)| \leq C\delta^{-2}$. We define the operator $K_\varepsilon = K_{\varepsilon, \delta} : L^2(\Omega_T) \rightarrow C_0^\infty(\Omega_T)$ by

$$K_\varepsilon(f)(x, t) = S_\varepsilon(\eta_1 \eta_2 f)(x, t). \quad (3.8)$$

Lemma 3.5. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $0 < T < \infty$. Let $u_\varepsilon, u_0 \in L^2(0, T; H^1(\Omega))$ be weak solutions of (1.4) and (1.6), respectively, for some $F \in L^2(\Omega_T)$. We further assume that $u_0 \in L^2(0, T; H^2(\Omega))$ and $\partial_t u_0 \in L^2(\Omega_T)$. Let w_ε be defined by (2.14), where the operator K_ε is given by (3.8). Then for any $\psi \in L^2(0, T; H_0^1(\Omega))$,*

$$\begin{aligned} &\left| \int_0^T \langle (\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon, \psi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} dt \right| \\ &\leq C \left\{ \|u_0\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T, 3\delta})} \right\} \\ &\quad \cdot \left\{ \varepsilon \|\nabla \psi\|_{L^2(\Omega_T)} + \varepsilon^{1/2} \|\nabla \psi\|_{L^2(\Omega_{T, 3\delta})} \right\}, \end{aligned} \quad (3.9)$$

where

$$\Omega_{T, \delta} = (\{x \in \Omega : \text{dist}(x, \partial\Omega) \leq \delta\} \times (0, T)) \cup (\Omega \times (0, \delta^2)) \cup (\Omega \times (T - \delta^2, T)), \quad (3.10)$$

and $C > 0$ depends at most on d, m, μ, T and Ω .

Proof. Using Theorem 2.2, it is not hard to see that the l.h.s. of (3.9) is bounded by

$$\begin{aligned}
& C \iint_{\Omega_T} |\nabla u_0 - K_\varepsilon(\nabla u_0)| |\nabla \psi| \\
& + C\varepsilon \iint_{\Omega_T} \left\{ |\chi^\varepsilon| + |\phi^\varepsilon| + |(\nabla \phi)^\varepsilon| \right\} |\nabla K_\varepsilon(\nabla u_0)| |\nabla \psi| \\
& + C\varepsilon^2 \iint_{\Omega_T} |\phi^\varepsilon| \left\{ |\partial_t K_\varepsilon(\nabla u_0)| + |\nabla^2 K_\varepsilon(\nabla u_0)| \right\} |\nabla \psi| \\
& = I_1 + I_2 + I_3,
\end{aligned} \tag{3.11}$$

where C depends only on d, m and μ . To estimate I_2 , we note that

$$\nabla K_\varepsilon(\nabla u_0) = \nabla S_\varepsilon(\eta_1 \eta_2(\nabla u_0)) = S_\varepsilon(\nabla(\eta_1 \eta_2)(\nabla u_0)) + S_\varepsilon(\eta_1 \eta_2(\nabla^2 u_0)). \tag{3.12}$$

It follows by the Cauchy inequality and Lemma 3.3 that

$$\begin{aligned}
I_2 & \leq C\varepsilon \left(\iint_{\Omega_T} \left\{ |\chi^\varepsilon| + |\phi^\varepsilon| + |(\nabla \phi)^\varepsilon| \right\} S_\varepsilon(\nabla(\eta_1 \eta_2)(\nabla u_0))^2 \right)^{1/2} \left(\iint_{\Omega_{T,3\delta}} |\nabla \psi|^2 \right)^{1/2} \\
& + C\varepsilon \left(\iint_{\Omega_T} \left\{ |\chi^\varepsilon| + |\phi^\varepsilon| + |(\nabla \phi)^\varepsilon| \right\} S_\varepsilon(\eta_1 \eta_2(\nabla^2 u_0))^2 \right)^{1/2} \left(\iint_{\Omega_T} |\nabla \psi|^2 \right)^{1/2} \\
& \leq C \left(\iint_{\Omega_{T,3\delta}} |\nabla u_0|^2 \right)^{1/2} \left(\iint_{\Omega_{T,3\delta}} |\nabla \psi|^2 \right)^{1/2} \\
& + C\varepsilon \left(\iint_{\Omega_T} |\nabla^2 u_0|^2 \right)^{1/2} \left(\iint_{\Omega_T} |\nabla \psi|^2 \right)^{1/2},
\end{aligned}$$

where we also have used the observation that $S_\varepsilon(\nabla(\eta_1 \eta_2)(\nabla u_0))$ is supported in $\Omega_{T,3\delta}$. This shows that I_2 is bounded by the r.h.s. of (3.9).

Next, to handle the term I_3 , we note that

$$\begin{aligned}
\partial_t K_\varepsilon(\nabla u_0) & = \partial_t S_\varepsilon(\eta_1 \eta_2(\nabla u_0)) = S_\varepsilon(\partial_t(\eta_1 \eta_2) \nabla u_0) + S_\varepsilon(\eta_1 \eta_2(\nabla \partial_t u_0)) \\
& = S_\varepsilon(\partial_t(\eta_1 \eta_2) \nabla u_0) + \nabla S_\varepsilon(\eta_1 \eta_2(\partial_t u_0)) - S_\varepsilon(\nabla(\eta_1 \eta_2)(\partial_t u_0)),
\end{aligned}$$

and

$$\nabla^2 K_\varepsilon(\nabla u_0) = \nabla S_\varepsilon(\nabla(\eta_1 \eta_2)(\nabla u_0)) + \nabla S_\varepsilon(\eta_1 \eta_2(\nabla^2 u_0)).$$

As in the case of I_2 , by the Cauchy inequality and Remark 3.4, this gives

$$\begin{aligned}
I_3 & \leq C \left(\iint_{\Omega_{T,3\delta}} |\nabla u_0|^2 \right)^{1/2} \left(\iint_{\Omega_{T,3\delta}} |\nabla \psi|^2 \right)^{1/2} \\
& + C\varepsilon \left(\iint_{\Omega_T} |\partial_t u_0|^2 \right)^{1/2} \left(\iint_{\Omega_T} |\nabla \psi|^2 \right)^{1/2} \\
& + C\varepsilon \left(\iint_{\Omega_T} |\nabla^2 u_0|^2 \right)^{1/2} \left(\iint_{\Omega_T} |\nabla \psi|^2 \right)^{1/2},
\end{aligned}$$

which is bounded by the r.h.s. of (3.9).

Finally, to estimate I_1 , we observe that

$$\begin{aligned}
I_1 &\leq C \iint_{\Omega_{T,2\delta}} \left\{ |\nabla u_0| + S_\varepsilon(\eta_1 \eta_2 |\nabla u_0|) \right\} |\nabla \psi| + C \iint_{\Omega_T \setminus \Omega_{T,2\delta}} |(\nabla u_0 - S_\varepsilon(\nabla u_0))| |\nabla \psi| \\
&\leq C \left(\iint_{\Omega_{T,3\delta}} |\nabla u_0|^2 \right)^{1/2} \left(\iint_{\Omega_{T,3\delta}} |\nabla \psi|^2 \right)^{1/2} \\
&\quad + C \left(\iint_{\Omega \setminus \Omega_{T,2\delta}} |\nabla u_0 - S_\varepsilon(\nabla u_0)|^2 \right)^{1/2} \left(\iint_{\Omega_T} |\nabla \psi|^2 \right)^{1/2}.
\end{aligned} \tag{3.13}$$

To treat the second term in the r.h.s. of (3.13), we extend u_0 to a function \tilde{u}_0 in \mathbb{R}^{d+1} such that

$$\left(\iint_{\mathbb{R}^{d+1}} |\nabla^2 \tilde{u}_0|^2 \right)^{1/2} + \left(\iint_{\mathbb{R}^{d+1}} |\partial_t \tilde{u}_0|^2 \right)^{1/2} \leq C \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} \right\},$$

using the Calderón's extension theorem. It follows that

$$\begin{aligned}
\left(\iint_{\Omega \setminus \Omega_{T,2\delta}} |\nabla u_0 - S_\varepsilon(\nabla u_0)|^2 \right)^{1/2} &\leq \left(\iint_{\mathbb{R}^{d+1}} |\nabla \tilde{u}_0 - S_\varepsilon(\nabla \tilde{u}_0)|^2 \right)^{1/2} \\
&\leq C\varepsilon \left\{ \|\nabla^2 \tilde{u}_0\|_{L^2(\mathbb{R}^{d+1})} + \|\partial_t \tilde{u}_0\|_{L^2(\mathbb{R}^{d+1})} \right\} \\
&\leq C\varepsilon \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} \right\},
\end{aligned}$$

where we have used Lemma 3.2 for the second inequality. As a result, we see that I_1 is also bounded by the r.h.s. of (3.9). This completes the proof. \square

Remark 3.6. Let $\Omega^\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \delta\}$. Then

$$\int_{\Omega^\delta} |\nabla u_0|^2 \leq C\delta \|\nabla u_0\|_{H^1(\Omega)}^2 \tag{3.14}$$

(see e.g. [16] for a proof). It follows that

$$\begin{aligned}
\|\nabla u_0\|_{L^2(\Omega_{T,3\delta})} &\leq \left(\int_0^T \int_{\Omega^{3\delta}} |\nabla u_0|^2 \right)^{1/2} + \left(\int_0^{c\varepsilon^2} \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} + \left(\int_{T-c\varepsilon^2}^T \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \\
&\leq C\varepsilon^{1/2} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \right\}.
\end{aligned}$$

The next theorem provides an $O(\sqrt{\varepsilon})$ error estimate in $L^2(0,T;H_0^1(\Omega))$ for the initial-Dirichlet problem (1.4).

Theorem 3.7. *Let w_ε be defined by (2.14). Under the same assumptions as in Lemma 3.5, we have*

$$\begin{aligned} & \|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \\ & \leq C\sqrt{\varepsilon} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \right\}, \end{aligned} \quad (3.15)$$

where C depends at most on d, m, μ, T and Ω .

Proof. Note that $w_\varepsilon \in L^2(0, T; H_0^1(\Omega))$ and $w_\varepsilon = 0$ on $\Omega \times \{t = 0\}$. It follows that

$$\begin{aligned} \mu \iint_{\Omega_T} |\nabla w_\varepsilon|^2 & \leq \int_0^T \langle (\partial_t + \mathcal{L}_\varepsilon) w_\varepsilon, w_\varepsilon \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ & \leq C\sqrt{\varepsilon} \|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,60\varepsilon})} \right\}, \end{aligned}$$

where we have used Lemma 3.5 for the last step. This, together with Remark 3.6, gives (3.15). \square

Next we consider the initial-Neumann problem (1.5).

Lemma 3.8. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $0 < T < \infty$. Let $u_\varepsilon, u_0 \in L^2(0, T; H^1(\Omega))$ be weak solutions of the initial-Neumann problems (1.5) and (1.7), respectively, for some $F \in L^2(\Omega_T)$. We further assume that $u_0 \in L^2(0, T; H^2(\Omega))$ and that $\partial_t u_0 \in L^2(\Omega_T)$. Let w_ε be defined by (2.14), where the operator K_ε is given by (3.8). Then for any $\psi \in L^2(0, T; H^1(\Omega))$,*

$$\begin{aligned} & \left| \int_0^T \langle \partial_t w_\varepsilon, \psi \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \right| \\ & \leq C \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,3\delta})} \right\} \\ & \quad \cdot \left\{ \varepsilon \|\nabla \psi\|_{L^2(\Omega_T)} + \varepsilon^{1/2} \|\nabla \psi\|_{L^2(\Omega_{T,3\delta})} \right\}, \end{aligned} \quad (3.16)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^1(\Omega)$ and its dual. The constant $C > 0$ depends at most on d, m, μ, T and Ω .

Proof. This follows from Theorem 2.3 by the same argument as in the proof of Lemma 3.5. \square

Theorem 3.9. *Let w_ε be defined by (2.14). Under the same assumptions as in Lemma 3.8, we have*

$$\begin{aligned} & \|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \\ & \leq C\sqrt{\varepsilon} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \right\}, \end{aligned} \quad (3.17)$$

where C depends at most on d, m, μ, T and Ω .

Proof. As in the proof of Theorem 3.7, this follows from Lemma 3.8 by letting $\psi = w_\varepsilon$. \square

Remark 3.10. In the case of $u_\varepsilon = u_0 = 0$ or $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \frac{\partial u_0}{\partial \nu_0} = 0$ on $\partial\Omega \times (0, T)$, we may bound the third term in the r.h.s. of (3.17) as follows. Note that

$$\int_{\Omega} \widehat{A} \nabla u_0 \cdot \nabla u_0 = - \int_{\Omega} \partial_t u_0 \cdot u_0 + \int_{\Omega} F \cdot u_0. \quad (3.18)$$

It follows that

$$\begin{aligned} \mu \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 &\leq \int_{t-\varepsilon^2}^t \int_{\Omega} |\partial_t u_0| |u_0| + \int_{t-\varepsilon^2}^t \int_{\Omega} |F| |u_0| \\ &\leq \left\{ \|\partial_t u_0\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} \right\} \left(\int_{t-\varepsilon^2}^t \int_{\Omega} |u_0|^2 \right)^{1/2} \\ &\leq \varepsilon \left\{ \|\partial_t u_0\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} \right\} \sup_{0 < t < T} \|u_0(\cdot, t)\|_{L^2(\Omega)}. \end{aligned}$$

This, together with the standard energy estimates, gives

$$\sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \leq C \left\{ \|\partial_t u_0\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} + \|h\|_{L^2(\Omega)} \right\}, \quad (3.19)$$

where C depends only on d, m, μ and Ω . As a result, for both the initial-Dirichlet problem (1.4) and the initial-Neumann problem (1.5), if $g = 0$ on $\partial\Omega \times (0, T)$, then

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \leq C \sqrt{\varepsilon} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|F\|_{L^2(\Omega_T)} + \|h\|_{L^2(\Omega)} \right\}, \quad (3.20)$$

where we have used the fact

$$\|\partial_t u_0\|_{L^2(\Omega_T)} \leq C \left\{ \|\nabla^2 u_0\|_{L^2(\Omega_T)} + \|F\|_{L^2(\Omega_T)} \right\}.$$

In particular, if Ω is $C^{1,1}$, $g = 0$ on $\partial\Omega \times (0, T)$ and $h = 0$ on Ω , then

$$\|\nabla w_\varepsilon\|_{L^2(\Omega_T)} \leq C \sqrt{\varepsilon} \|F\|_{L^2(\Omega_T)}. \quad (3.21)$$

To see this, we use the well-known estimate

$$\|u_0\|_{L^2(0,T;H^2(\Omega))} \leq C \|F\|_{L^2(\Omega_T)}, \quad (3.22)$$

which may be proved by using the partial Fourier transform in the t variable and reducing the problem to the H^2 estimate for the elliptic operator \mathcal{L}_0 in $C^{1,1}$ domains. We also note that in the case that $g = 0$ on $\partial\Omega \times (0, T)$ and $h \in H^1(\Omega; \mathbb{R}^m)$, if $\mathcal{L}_0^* = \mathcal{L}_0$ and Ω is $C^{1,1}$, then

$$\|u_0\|_{L^2(0,T;H^2(\Omega))} \leq C \left\{ \|F\|_{L^2(\Omega_T)} + \|h\|_{H^1(\Omega)} \right\}. \quad (3.23)$$

This may be proved by using integration by parts as well as H^2 estimates for \mathcal{L}_0 [13].

4 Proof of Theorems 1.1 and 1.2

In this section we study the convergence rates in $L^2(\Omega_T)$ and give the proof of Theorems 1.1 and 1.2. Throughout the section we will assume that Ω is a bounded $C^{1,1}$ domain in \mathbb{R}^d .

We first consider the initial-Dirichlet problem. Let A^* denote the adjoint of A ; i.e., $A^* = (a_{ij}^{*\alpha\beta})$ with $a_{ij}^{*\alpha\beta}(y, s) = a_{ji}^{\beta\alpha}(y, s)$. For $G \in L^2(\Omega_T)$, let v_ε be the weak solution to

$$\begin{cases} (-\partial_t + \mathcal{L}_\varepsilon^*)v_\varepsilon = G & \text{in } \Omega \times (0, T), \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0, T), \\ v_\varepsilon = 0 & \text{on } \Omega \times \{t = T\}, \end{cases} \quad (4.1)$$

where $\mathcal{L}_\varepsilon^* = -\operatorname{div}(A^{*\varepsilon}(x, t)\nabla)$ denotes the adjoint of \mathcal{L}_ε , and v_0 the weak solution to

$$\begin{cases} (-\partial_t + \mathcal{L}_0^*)v_0 = G & \text{in } \Omega \times (0, T), \\ v_0 = 0 & \text{on } \partial\Omega \times (0, T), \\ v_0 = 0 & \text{on } \Omega \times \{t = T\}, \end{cases} \quad (4.2)$$

where $\mathcal{L}_0^* = -\operatorname{div}(\hat{A}^*\nabla)$. Observe that $v_\varepsilon(x, T - t)$ and $v_0(x, T - t)$ are solutions of the initial-Dirichlet problems of (1.4) and (1.6), respectively, with coefficient matrix $A(x/\varepsilon, t/\varepsilon^2)$ replaced by $A^*(x/\varepsilon, (T - t)/\varepsilon^2)$, and with $g = 0$ and $h = 0$. Also note that $A^*(y, T - s)$ satisfies the same ellipticity and periodicity conditions as $A(y, s)$.

Lemma 4.1. *Let v_0 be the weak solution to (4.2). Then*

$$\|\nabla v_0\|_{L^2(\Omega_T)} + \delta^{-1/2}\|\nabla v_0\|_{L^2(\Omega_{T,\delta})} \leq C\|G\|_{L^2(\Omega_T)}, \quad (4.3)$$

where $\delta \in (2\varepsilon, 20\varepsilon)$ and C depends at most on d, m, μ, T and Ω .

Proof. The estimate for $\|\nabla v_0\|_{L^2(\Omega_T)}$ follows directly from the energy estimate, while the estimate for $\delta^{-1/2}\|\nabla v_0\|_{L^2(\Omega_{T,\delta})}$ is proved in Remarks 3.6 and 3.10. \square

Let

$$\begin{aligned} z_\varepsilon(x, t) = v_\varepsilon(x, T - t) - v_0(x, T - t) - \varepsilon \chi_{T,j}^{*\varepsilon} S_\varepsilon \left(\tilde{\eta}(x, t) \frac{\partial v_0}{\partial x_j}(x, T - t) \right) \\ - \varepsilon^2 \phi_{T,(d+1)ij}^{*\varepsilon} \frac{\partial}{\partial x_i} S_\varepsilon \left(\tilde{\eta}(x, t) \frac{\partial v_0}{\partial x_j}(x, T - t) \right), \end{aligned} \quad (4.4)$$

where χ_T^* and ϕ_T^* denote the correctors and dual correctors, respectively, for the family of parabolic operators $\partial_t + \operatorname{div}(A^*(x/\varepsilon, (T - t)/\varepsilon^2)\nabla)$, $\varepsilon > 0$. The cut-off function $\tilde{\eta}$ in (4.4) is chosen so that $\tilde{\eta}(x, t) = 0$ if $(x, t) \in \Omega_{T,10\varepsilon}$, $\tilde{\eta}(x, t) = 1$ if $(x, t) \in \Omega_T \setminus \Omega_{T,15\varepsilon}$, $|\nabla \tilde{\eta}| \leq C\varepsilon^{-1}$ and $|\partial_t \tilde{\eta}| \leq C\varepsilon^{-2}$.

Lemma 4.2. *Let z_ε be defined by (4.4). Then*

$$\|\nabla z_\varepsilon\|_{L^2(\Omega_T)} \leq C\sqrt{\varepsilon}\|G\|_{L^2(\Omega_T)}, \quad (4.5)$$

where C depends at most on d, m, μ, T and Ω .

Proof. Since $A^*(y, T-s)$ satisfies the same ellipticity and periodicity conditions as $A(y, s)$ and Ω is $C^{1,1}$, this follows from the estimate (3.21). \square

We are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $u_\varepsilon \in L^2(0, T; H^1(\Omega))$ and $u_0 \in L^2(0, T; H^2(\Omega))$ be solutions of (1.4) and (1.6), respectively. Let $G \in L^2(\Omega_T)$. By duality it suffices to show that

$$\begin{aligned} & \left| \iint_{\Omega_T} (u_\varepsilon - u_0) \cdot G \right| \\ & \leq C\varepsilon \|G\|_{L^2(\Omega_T)} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \right\}. \end{aligned} \quad (4.6)$$

Let w_ε be defined by (2.14), with $\delta = 2\varepsilon$. Since

$$\|\chi^\varepsilon K_\varepsilon(\nabla u_0)\|_{L^2(\Omega_T)} + \varepsilon \|\phi^\varepsilon \nabla K_\varepsilon(\nabla u_0)\|_{L^2(\Omega_T)} \leq C \|\nabla u_0\|_{L^2(\Omega_T)},$$

we only need to prove that $\left| \iint_{\Omega_T} w_\varepsilon \cdot G \right|$ is bounded by the r.h.s. of (4.6).

To this end we write

$$\begin{aligned} \iint_{\Omega_T} w_\varepsilon \cdot G &= \int_0^T \langle \partial_t w_\varepsilon, v_\varepsilon \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla v_\varepsilon \\ &= \left\{ \int_0^T \langle \partial_t w_\varepsilon, z_\varepsilon(\cdot, T-t) \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla z_\varepsilon(x, T-t) \right\} \\ &\quad + \left\{ \int_0^T \langle \partial_t w_\varepsilon, v_0 \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla v_0 \right\} \\ &\quad + \left\{ \int_0^T \langle \partial_t w_\varepsilon, v_\varepsilon - v_0 - z_\varepsilon(\cdot, T-t) \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \{v_\varepsilon - v_0 - z_\varepsilon(\cdot, T-t)\} \right\} \\ &= J_1 + J_2 + J_3, \end{aligned} \quad (4.7)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H_0^1(\Omega)$ and its dual $H^{-1}(\Omega)$. We shall use Lemma 3.5 to bound J_1 , J_2 and J_3 .

For the term J_1 , it follows by Lemma 3.5 that

$$\begin{aligned} |J_1| &\leq C\sqrt{\varepsilon} \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,6\varepsilon})} \right\} \|\nabla z_\varepsilon\|_{L^2(\Omega_T)} \\ &\leq C\varepsilon \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,6\varepsilon})} \right\} \|G\|_{L^2(\Omega_T)}, \end{aligned} \quad (4.8)$$

where we have used Lemma 4.2 for the last step.

Next, for J_2 , we obtain

$$\begin{aligned} |J_2| &\leq C \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,6\varepsilon})} \right\} \\ &\quad \cdot \left\{ \varepsilon \|\nabla v_0\|_{L^2(\Omega)} + \varepsilon^{1/2} \|\nabla v_0\|_{L^2(\Omega_{T,6\varepsilon})} \right\} \\ &\leq C\varepsilon \left\{ \|u_0\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T,6\varepsilon})} \right\} \|G\|_{L^2(\Omega_T)}, \end{aligned} \quad (4.9)$$

where we have used Lemma 4.1 for the last inequality.

To estimate J_3 , we note that $v_\varepsilon - v_0 - z_\varepsilon(x, T - t)$ is supported in $\Omega_T \setminus \Omega_{T, 10\varepsilon}$ and in view of (4.4) and Lemmas 3.1 and 3.3,

$$\|\nabla(v_\varepsilon - v_0 - z_\varepsilon(x, T - t))\|_{L^2(\Omega_T)} \leq C\|\nabla v_0\|_{L^2(\Omega_T)} \leq C\|G\|_{L^2(\Omega_T)}.$$

It follows by Lemma 3.5 that

$$|J_3| \leq C\varepsilon \left\{ \|u_0\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T, 6\varepsilon})} \right\} \|G\|_{L^2(\Omega_T)}.$$

This, together with (4.8) and (4.9), shows that

$$\begin{aligned} & \left| \iint_{\Omega_T} w_\varepsilon \cdot G \right| \\ & \leq C\varepsilon \left\{ \|u_0\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \varepsilon^{-1/2} \|\nabla u_0\|_{L^2(\Omega_{T, 6\varepsilon})} \right\} \|G\|_{L^2(\Omega_T)} \\ & \leq C\varepsilon \left\{ \|u_0\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t u_0\|_{L^2(\Omega_T)} + \sup_{\varepsilon^2 < t < T} \left(\frac{1}{\varepsilon} \int_{t-\varepsilon^2}^t \int_{\Omega} |\nabla u_0|^2 \right)^{1/2} \right\} \|G\|_{L^2(\Omega_T)}, \end{aligned}$$

which completes the proof. \square

Finally, we give the proof of Theorem 1.2

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar to that of Theorem 1.1. Indeed, let $u_\varepsilon \in L^2(0, T; H^1(\Omega))$ and $u_0 \in L^2(0, T; H^2(\Omega))$ be solutions of (1.5) and (1.7), respectively. Let w_ε be defined as in (2.14), with $\delta = 2\varepsilon$. To estimate $\|u_\varepsilon - u_0\|_{L^2(\Omega_T)}$, we consider $\iint_{\Omega_T} w_\varepsilon \cdot G$, where $G \in L^2(\Omega_T)$. Let v_ε be the weak solution to

$$\begin{cases} (-\partial_t + \mathcal{L}_\varepsilon^*)v_\varepsilon = G & \text{in } \Omega \times (0, T), \\ \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon^*} = 0 & \text{on } \partial\Omega \times (0, T), \\ v_\varepsilon = 0 & \text{on } \Omega \times \{t = T\}, \end{cases} \quad (4.10)$$

and v_0 the weak solution to

$$\begin{cases} (-\partial_t + \mathcal{L}_0^*)v_0 = G & \text{in } \Omega \times (0, T), \\ \frac{\partial v_0}{\partial \nu_0^*} = 0 & \text{on } \partial\Omega \times (0, T), \\ v_0 = 0 & \text{on } \Omega \times \{t = T\}, \end{cases} \quad (4.11)$$

where $\frac{\partial v_\varepsilon}{\partial \nu_\varepsilon^*}$ and $\frac{\partial v_0}{\partial \nu_0^*}$ denote the conormal derivatives associated with the operators $\mathcal{L}_\varepsilon^*$ and \mathcal{L}_0^* , respectively. Let z_ε be defined as before. Note that estimates in Lemmas 4.1 and 4.2 continue to hold. Moreover, by (4.10), we have

$$\iint_{\Omega_T} w_\varepsilon \cdot G = \int_0^T \langle \partial_t w_\varepsilon, v_\varepsilon \rangle + \iint_{\Omega_T} A^\varepsilon \nabla w_\varepsilon \cdot \nabla v_\varepsilon,$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^1(\Omega)$ and its dual. With Lemma 3.8 at our disposal, the rest of the proof is exactly the same as that of Theorem 1.1. We omit the details. \square

References

- [1] S.N. Armstrong and Z. Shen, *Lipschitz estimates in almost-periodic homogenization*, Comm. Pure Appl. Math. (to appear).
- [2] S.N. Armstrong and C.K. Smart, *Quantitative stochastic homogenization of convex integral functionals*, Ann. Sci. Éc. Norm. Supér (to appear).
- [3] M. Avellaneda and F. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math. **40** (1987), 803–847.
- [4] A. Bensoussan, J.-L. Lions, and G.C. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, North Holland, 1978.
- [5] J. Geng and Z. Shen, *Uniform regularity estimates in parabolic homogenization*, Indiana Univ. Math. J. **64** (2015), 697–733.
- [6] G. Griso, *Error estimate and unfolding for periodic homogenization*, Asymptot. Anal. **40** (2004), 269–286.
- [7] ———, *Interior error estimate for periodic homogenization*, Anal. Appl. (Singap.) **4** (2006), no. 1, 61–79.
- [8] S. Gu, *Convergence rates in homogenization of Stokes systems*, arXiv:1508.04203 (2015).
- [9] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, 1994.
- [10] C. Kenig, F. Lin, and Z. Shen, *Convergence rates in L^2 for elliptic homogenization problems*, Arch. Rational Mech. Anal. **203** (2012), no. 3, 1009–1036.
- [11] ———, *Estimates of eigenvalues and eigenfunctions in periodic homogenization*, J. Eur. Math. Soc. **15** (2013), 1901–1925.
- [12] ———, *Periodic homogenization of Green and Neumann functions*, Comm. Pure Appl. Math. **67** (2014), 1219–1262.
- [13] Q.A. Ladyzenskaja, Solonnikov V.A., and U.N. Uralceva, *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Soc., 1968.
- [14] D. Onofrei and B. Vernescu, *Error estimates for periodic homogenization with non-smooth coefficients*, Asymptot. Anal. **54** (2007), 103–123.
- [15] Z. Shen, *Boundary estimates in elliptic homogenization*, arXiv:1505.02525 (2015).
- [16] Z. Shen and J. Zhuge, *Convergence rates in periodic homogenization of systems of elasticity with mixed boundary conditions*, arXiv:1512.00823 (2015).
- [17] T.A. Suslina, *On the averaging of periodic parabolic systems*, Funct. Anal. Appl. **38** (2004), 309–312.

- [18] ———, *Homogenization of the elliptic Dirichlet problem: operator error estimates in L_2* , *Mathematika* **59** (2013), no. 2, 463–476.
- [19] ———, *Homogenization of the Neumann problem for elliptic systems with periodic coefficients*, *SIAM J. Math. Anal.* **45** (2013), no. 6, 3453–3493.
- [20] ———, *Homogenization of solutions of initial boundary value problems for parabolic systems*, *Func. Anal. Appl.* **49** (2015), 72–76.
- [21] V.V. Zhikov and S.E. Pastukhova, *Estimates of homogenization for a parabolic equation with periodic coefficients*, *Russ. J. Math. Phys.* **13** (2006), 224–237.

Jun Geng, School of Mathematics and Statistics, Lanzhou University, Lanzhou, P.R. China.
E-mail: gengjun@lzu.edu.cn

Zhongwei Shen, Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506, USA.
E-mail: zshen2@uky.edu